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Turing structures and stability for the 1-D Lengyel–Epstein system

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Abstract This paper continues the analysis on the Lengyel–Epstein reaction-diffusion system of the chlorite-iodide-malonic acid-starch (CIMA) reaction for the rich Turing structures. The steady state structures, especially the double bifurcation one, and their stability and multiplicity are studied by the use of Lyapunov–Schmidt reduction technique and singularity theory. Numerical simulations are presented to support our theoretical studies. The results show that the richer stationary Turing patterns heavily rely both on the size of the reactor and on the effective diffusion rate in the CIMA reaction.

Keywords Lengyel–Epstein system · Turing bifurcation · Stability · Lyapunov–Schmidt procedure · Normal form

1 Introduction

Pattern formation is a classical issue of interest in many branches of nonlinear science, such as physics, chemistry, biology, and social disciplines. One of the most well-known mechanism for pattern formation is the diffusion-driven or Turing instability suggested in Turing's seminal paper "The Chemical Basis of Morphogenesis" [1], which shows that diffusion can destabilize a homogeneous steady state and result in the formation of nonhomogeneous stationary structures. After then the mechanism has been utilized to explain the pattern formation in fields ranging from economics [2], biology and chemistry [3] to astrophysics [4,5], where the most fruitful area of investigation is likely to be biology.

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Despite the profound impact of Turing's idea on theoretical developments in pattern formation, the Turing's prediction was not verified experimentally until 1990, nearly 40 years after the original theory, by De Kepper et al. on the chlorite-iodide-malonic acid-starch (CIMA) reaction in an open unstirred gel reactor [6]. They observed the formation of spotty structure which represents a significant breakthrough for one of the most fundamental ideas in morphogenesis and biological pattern formation. Subsequently, considering that three of the five reactants remain nearly constant in the CIMA reaction, Lengyel and Epstein [7,8] developed a simple two-variable model, i.e. so-called Lengyel–Epstein model, which takes the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + a - u - \dfrac{4uv}{1+u^2}, & x \in \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = \sigma \left[c\Delta v + b\left(u - \dfrac{uv}{1+u^2} \right) \right], & x \in \Omega,\ t > 0, \\ \partial u/\partial \nu = 0,\ \partial v/\partial \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n, with a smooth boundary $\partial\Omega$; $u(x,t)$ and $v(x,t)$ denote the dimensionless concentrations of iodide (I^-) and chlorite (ClO_2^-), respectively; a and b are the parameters related to the feed concentrations; c is the ratio of the diffusion coefficients; $\sigma > 1$ is a rescaling parameter depending on the concentration of the starch, enlarging the effective diffusion ratio to σc. It is always assumed that all parameters a, b, c and σ are positive constants.

After the successful evidence [6], a number of important experimental and numerical works [9–13] and rigorous mathematical investigations [14–18] focus on the system (1.1). When the spatial domain is one-dimensional, Yi et al. [14], choosing b as the bifurcation parameter, performed a detailed Hopf bifurcation analysis for both the ODE and PDE models, and investigated the direction of the Hopf bifurcation and the stability of the bifurcating spatially homogeneous periodic solutions. In [15], they further considered the global asymptotical behavior of constant steady state when the feeding rate of iodide a is small, and showed that for small spatial domains, all solutions eventually converge to a spatially homogeneous and time-periodic solution. In [16], taking the feeding rate a of iodide as the bifurcation parameter, the authors proved that the PDE system (1.1) undergoes a sequence of bifurcations generating spatially nonhomogeneous time-periodic solutions and steady state solutions, which strongly suggested the richness of spatiotemporal dynamics.

Certainly, more original and systematic works on mathematical aspects were proposed by Ni and Tang [17,18]. In [17], they studied the non-existence of Turing patterns and the Turing instability, and showed that Turing structures form only if the parameter a (related to the feed concentrations), the size of the reactor Ω (reflected by its first eigenvalue), or the "effective" diffusion rate $d = c/b$ are suitable large. Furthermore, they argued that if the parameter a lies in a suitable range, then the system possesses non-constant steady states for large d. For the better description of the structures, in the one-dimensional case, they [18] further considered the global bifurcation of the non-constant steady states arising from the simple bifurcation (i.e. the case $d_j \neq d_k$ in [18]) by taking the "effective" diffusion rate d as bifurcation parameter.

This paper continues the analytic works of [18] with the goal of revealing a new or rich solution structure and achieving a deeper understanding of the Turing patterns operating in the system (1.1). We still view the effective diffusion rate d as the bifurcation parameter, and maintain the basic hypothesis on the system parameters, i.e. condition (H) below, which results in that no spatiotemporal pattern exists for the bifurcation parameter d. Thus, our main purpose is only to study Turing structures, especially bifurcating from the double eigenvalue (i.e. the case $d_j = d_k$) by using Lyapunov–Schmidt technique and singularity theory [19], and further determine the stability and multiplicity of the bifurcating non-constant steady state solutions, in particular the simple bifurcation solutions obtained in [18]. The results enrich and perfect the earlier works of [18] in order to seeking a complete mathematical investigations of the Turing patterns for the Lengyel–Epstein system (1.1), and are of practical significance for the researches on pattern formation in complex reaction-diffusion systems. We believe that our theoretical studies of spatially nonhomogeneous steady states are new advance, and until now little or no results is known about the effect of diffusion on such double bifurcation structure.

To complement the previous works, we firstly recall some results of [17,18] in the next section, and present the basic assumption (H) on the system parameters. Because no Hopf bifurcation occurs based on such condition (H) for the bifurcation parameter d, we just go on with the discussion of Turing bifurcation in Sect. 3, whose key point rests on the double bifurcation. In Sect. 4, we further consider the stability and the bifurcation direction of the bifurcating non-constant steady states, especially stationary structures illustrated in [18], deriving a more detailed description of the Turing patterns for the Lengyel–Epstein system. Finally, Sect. 5 is devoted to numerical simulations for confirming the analytic results of the previous sections.

2 Preliminary

In the present section, we describe the results in [17,18] for the later discussions. By introducing the new variable $d = c/b$, $\delta = \sigma b$ and $\alpha = a/5$, the system (1.1) in the one-dimensional interval $\Omega = (0, l)$ can be written as

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + 5\alpha - u - \dfrac{4uv}{1+u^2}, & x \in (0,l),\ t > 0, \\ \dfrac{\partial v}{\partial t} = \delta\left(d\dfrac{\partial^2 v}{\partial x^2} + u - \dfrac{uv}{1+u^2}\right), & x \in (0,l),\ t > 0, \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0, & x = 0, l,\ t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0, & x \in (0,l). \end{cases} \tag{2.1}$$

Clearly, system (1.1) and (2.1) both has a unique spatially homogeneous steady state $(u^*, v^*) = (\alpha, 1+\alpha^2)$. As in [17], we denote

$$f(u,v) = 5\alpha - u - \frac{4uv}{1+u^2}, \quad g(u,v) = u - \frac{uv}{1+u^2},$$
$$f_0 := f_u(u^*, v^*) = \frac{3\alpha^2 - 5}{1+\alpha^2} < 3, \quad f_1 := f_v(u^*, v^*) = -\frac{4\alpha}{1+\alpha^2} < 0,$$
$$g_0 := g_u(u^*, v^*) = \frac{2\alpha^2}{1+\alpha^2} > 0, \quad g_1 := g_v(u^*, v^*) = -\frac{\alpha}{1+\alpha^2} < 0,$$

and still maintain the basic hypothesis

$$\text{(H)} \quad 0 < 3\alpha^2 - 5 < \delta\alpha$$

which leads to that

(i) no Hopf bifurcation occurs based on the work [17] for taking d as the bifurcation parameter,
(ii) the system (1.1) is an activator-inhibitor model where it is commonly assumed that the inhibitor v inhibits the production of activator u, and u activates itself and the inhibitor.
(iii) the unique constant solution (u^*, v^*) of (1.1) is diffusion-free stable.

Let $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$ be the sequence of eigenvalues for the elliptic operator $-\Delta$ subject to the Neumann boundary condition on Ω, where each λ_i has multiplicity $m_i \geq 1$. If $\lambda_1 < f_0$, then we define $i_\alpha = i_\alpha(\alpha, \Omega)$ $(1 \leq i_\alpha < \infty)$ to be the largest positive integer such that $\lambda_i < f_0$ for $1 \leq i \leq i_\alpha$. Thus, we can set

$$\tilde{d} = \min_{1 \leq i \leq i_\alpha} d_i, \quad d_i = \frac{\alpha}{1+\alpha^2} \frac{\lambda_i + 5}{\lambda_i (f_0 - \lambda_i)}. \tag{2.2}$$

Then the non-existence of nonconstant steady states and Turing instability of (u^*, v^*) are demonstrated as follows.

Theorem 2.1 [17] *There is a constant* $d_0 = d_0(a, \lambda_1) > 0$ *such that the system* (1.1) *does not admit a nonconstant solution for* $0 < d < d_0$.

Lemma 2.2 [17] *Assume (H) hold. If* $\lambda_1 \geq f_0$*, or* $\lambda_1 < f_0$ *and* $0 < d < \tilde{d}$*, then the constant steady state* (u^*, v^*) *is asymptotically stable. If* $\lambda_1 < f_0$ *and* $d > \tilde{d}$*, then* (u^*, v^*) *is unstable, and hence Turing unstable.*

When the constant solution (u^*, v^*) becomes unstable, the Turing structures or the non-constant steady states are naturally concerned for $d > \tilde{d}$. Hence, in the one-dimensional interval $\Omega = (0, l)$, Jang et al. [18] give a detailed description of the Turing structures.

For the domain $\Omega = (0, l)$, it is well known that the elliptic operator $-\Delta$ subject to the Neumann boundary condition possesses eigenvalues $\lambda_j = (\pi j / l)^2$ $(j = 0, 1, 2, \ldots)$ whose corresponding normalized eigenfunctions are given by

$$\phi_j = \begin{cases} \frac{1}{\sqrt{l}}, & j = 0, \\ \sqrt{\frac{2}{l}} cos \frac{\pi j}{l} x, & j > 0. \end{cases}$$

Then the set $\{\phi_j\}$, $j = 0, 1, 2, \ldots$ forms a complete orthonormal basis in $L^2(0, l)$.

Theorem 2.3 [18] *Suppose j is a positive integer such that $\lambda_j < f_0$ and $d_j \neq d_k$ for any integer $k \neq j$. Then (d_j, u^*, v^*) is a bifurcation point of system (2.1) with respect to the curve (d, u^*, v^*), $d > 0$.*

There is a one-parameter family of non-trivial solutions $\Gamma(s) = (d(s), u(s), v(s))$ of the system (2.1) for $|s|$ sufficiently small, where $d(s), u(s), v(s)$ are continuous functions, $d(0) = d_j$ and

$$\begin{aligned} u_j(s) = u^* + s\phi_j + o(s),\ v_j(s) = v^* + sb_j\phi_j + o(s), \\ b_j = (\lambda_j - f_0)/f_1 > 0. \end{aligned} \tag{2.3}$$

The steady state solution set of system (2.1) consists of two curves (d, u^*, v^*) and $\Gamma(s)$ in a neighborhood of the bifurcation point (d_j, u^*, v^*).

Let J denote the closure of the non-trivial steady state solution set of system (2.1), and Γ_j the connected component of $J \cup \{(d_j, 0, 0)\}$ to which the trivial solution $\{(d_j, 0, 0)\}$ belongs. In a neighborhood of the bifurcation point the curve Γ_j is characterized by the eigenfunction ϕ_j. The detailed analysis on the bifurcating curve Γ_j far from the equilibrium is shown in the following theorem.

Theorem 2.4 [18] *Under the same assumption of Theorem 2.3, the projection of the bifurcation curve Γ_j on the $d-$axis contains (d_j, ∞).*

If $d > \tilde{d}$ and $d \neq d_k$ for any integer $k > 0$, then the system (2.1) possesses at least one non-constant positive solution.

We remark that there is no contradiction between Theorems 2.1 and 2.4 or 2.3 since it is easy to check that $\tilde{d} > d_0$. Moreover, Theorem 2.3 shows that all (d_j, u^*, v^*), $j = 1, 2, \ldots, i_\alpha$ are the simple bifurcation points on the basis of assumption every $d_j \neq d_k$ for any integer $k \neq j$. However, for some $d_j = d_k\,(j \neq k)$ in all (d_j, u^*, v^*), $j = 1, 2, \ldots, i_\alpha$, litter is known about the bifurcation solution. Therefore, our main contribution lies in the rigorous discussion for this case described in the next section.

3 Turing bifurcation when some $d_j = d_k$ for $j \neq k$

In this section, still using the effective diffusion rate d as the bifurcation parameter, we investigate the case that d_j, $j = 1, 2, \ldots, i_\alpha$ satisfy some $d_j = d_k$ for $j \neq k \in [1, i_\alpha]$. For this case, from (2.2) we can confirm that $d_j = d_k$ for $j \neq k$ if and only if

$$\alpha^2 = \frac{5 + \lambda^*_{jk}}{3 - \lambda^*_{jk}} > \frac{5}{3}, \quad \lambda^*_{jk} = \lambda_j + \lambda_k + \frac{1}{5}\lambda_j\lambda_k, \tag{3.1}$$

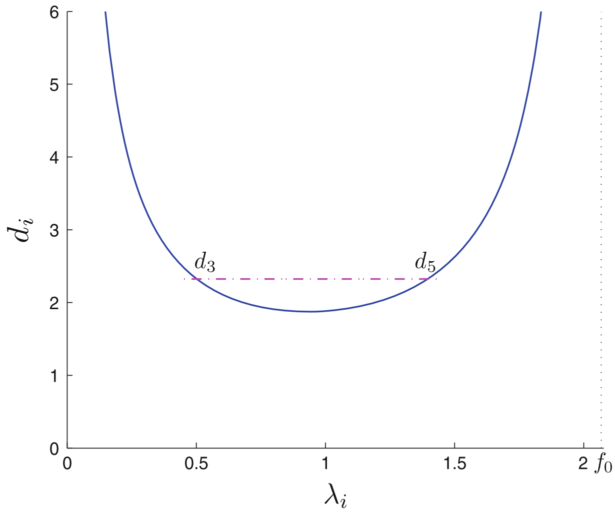

Fig. 1 The curve is described by (2.2) with $d_3 = d_5 = 2.3227$ for $j = 3, k = 5$ and $l = 13.3$, leading to $\alpha = 2.7034$ in (3.1), and $d_1 = 14.8809, d_2 = 4.1980, d_3 = 2.3227, d_4 = 1.8768, d_5 = 2.3227, d_6 = 39.8377$

which is the basic assumption for the present section and leads to $d_j = d_k = -\frac{5g_1}{\lambda_j \lambda_k}$, verified in Fig. 1. Moreover, we can obtain $\lambda^*_{jk} = f_0 < 3$, which means $\lambda_1 = (\pi/l)^2 < \iota^* < 1$, $\iota^* = 5(\sqrt{(j^2+k^2)^2 + 12j^2k^2/5} - j^2 - k^2)/(2j^2k^2)$, that is to say, the size of the reactor l must be greater than π for $d_j = d_k,\ j \neq k$. From (3.1), we also notice that in all $d_j,\ j = 1, 2, \ldots, i_\alpha$, greater than or equal to three quantities must be unequal, and there only exists a pair of quantities to be equal such as $d_j = d_k$ for some $j \neq k$.

For other d_j such that $d_j \neq d_k$ (any $k \neq j$), the bifurcation solution forms are same as (2.3) and the solutions can occur global behaviors according to [18] or Sect. 2. However, for such d_j satisfying $d_j = d_k\ (j \neq k)$, it is very necessary to point out that the classical bifurcation theory (for example [20]) used in [18] can not be applied because the hypothesis for the Crandall and Rabinowitz theorem is no longer satisfied. Therefore, the Lyapunov–Schmidt reduction technique and singularity theory [19] are the powerful tools to analyze how the bifurcation occurs.

Now we turn to discuss the d_j satisfying $d_j = d_k$ for $j \neq k$, which is our major concern. Without loss of generality, we always assume $j < k$ in the situation $d_j = d_k$ for $j \neq k$.

For analytical convenience, we introduce the new variables

$$\bar{u} = u - \alpha, \quad \bar{v} = v - 1 - \alpha^2.$$

In order to conserve notation, we drop the bars over the quantities and still denote $\bar{u}, \bar{v}$ as u, v. Taking these into consideration, system (2.1) can be written as

$$\begin{cases} \dfrac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = L\begin{pmatrix} u \\ v \end{pmatrix} + N(u, v), & x \in (0, l), \quad t > 0, \\ \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial x} = 0, & x = 0, l, \quad t > 0, \\ u(x, 0) = u_0(x) > 0, \; v(x, 0) = v_0(x) > 0, & x \in (0, l), \end{cases}$$

where the linear part

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \Delta + f_0 & f_1 \\ \delta g_0 & \delta(d\Delta + g_1) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \;\; \Delta = \frac{\partial^2}{\partial x^2} \tag{3.2}$$

and the nonlinear part

$$N(u, v) = \left[-\frac{(u + \alpha)(v + 1 + \alpha^2)}{1 + (u + \alpha)^2} + \alpha + \frac{1 - \alpha^2}{1 + \alpha^2} u + \frac{\alpha}{1 + \alpha^2} v \right]\begin{pmatrix} 4 \\ \delta \end{pmatrix}. \tag{3.3}$$

Let $X = \{(u, v) : u, v \in C^2[0, l], \; u' = v' = 0 \text{ at } x = 0, l\}$, $Y = C^0[0, l] \times C^0[0, l]$, and define the inner product of Y by

$$\langle U_1, U_2 \rangle = \langle u_1, u_2 \rangle_{L^2(0,l)} + \langle v_1, v_2 \rangle_{L^2(0,l)}, \;\; U_1 = (u_1, v_1), \; U_2 = (u_2, v_2) \in Y,$$

and the smooth mapping $F : X \times R \to Y$ by

$$F(w, \lambda) = Lw + N(w), \; w = (u, v)^\top, \; \lambda = d - d_j,$$

where Lw and $N(w)$ are respectively given by (3.2) and (3.3). It is obvious that $F(0, \lambda) = 0$. Therefore, for obtaining the non-constant steady states of (2.1), we only need to consider the non-zero solutions of the elliptic problem

$$F(w, \lambda) = 0, \quad x \in (0, l) \tag{3.4}$$

with the boundary condition

$$\frac{\partial w}{\partial x} = 0, \quad \text{at } x = 0, l.$$

Now we take λ instead of d as the main bifurcation parameter for the further discussions.

The linearized operator of system (3.4) evaluated at $(w, \lambda) = (0, 0)$ is

$$L_0 = \begin{pmatrix} \Delta + f_0 & f_1 \\ \delta g_0 & \delta(d_j\Delta + g_1) \end{pmatrix}, \;\; \Delta = \frac{\partial^2}{\partial x^2}.$$

Let $(\phi, \psi) \in N(L_0)$ and $\phi = \sum_{i=0}^{\infty} a_i \phi_i,\ \psi = \sum_{i=0}^{\infty} b_i \phi_i$. Then we have

$$\sum_{i=0}^{\infty} B_i \begin{pmatrix} a_i \\ b_i \end{pmatrix} \phi_i = 0, \quad B_i = \begin{pmatrix} f_0 - \lambda_i & f_1 \\ \delta g_0 & \delta(g_1 - d_j \lambda_i) \end{pmatrix}. \tag{3.5}$$

On the basis of $d_j = d_k$ for $j \neq k$, we obtain that

$$N(L_0) = \mathrm{span}\{\Phi_j, \Phi_k\}, \ \ N(L_0^*) = \mathrm{span}\{\Phi_j^*, \Phi_k^*\}, \tag{3.6}$$

where

$$\Phi_i = \begin{pmatrix} 1 \\ b_i \end{pmatrix} \phi_i, \ b_i = \frac{\lambda_i - f_0}{f_1} > 0,$$
$$\Phi_i^* = \frac{1}{1 + b_i b_i^*} \begin{pmatrix} 1 \\ b_i^* \end{pmatrix} \phi_i, \ b_i^* = \frac{\bar{b}_i^*}{\delta}, \ \bar{b}_i^* = \frac{\lambda_i - f_0}{g_0} < 0, \ i = j, k,$$

normalized so that $\langle \Phi_i, \Phi_m^* \rangle = \delta_{im}, i, m = j, k$. It is easy to verify that $1 + b_i b_i^* > 0,\ i = j, k$, which will be used in the later discussions.

Similarly, the other two eigenvalues of L_0 corresponding to d_j, d_k are, respectively,

$$\mu_j = f_0 - \lambda_j + \delta(g_1 - d_j \lambda_j) = \frac{\delta g_1 (5 + \lambda_k)(1 + b_j b_j^*)}{\lambda_k} < 0,$$
$$\mu_k = f_0 - \lambda_k + \delta(g_1 - d_k \lambda_k) = \frac{\delta g_1 (5 + \lambda_j)(1 + b_k b_k^*)}{\lambda_j} < 0,$$

with eigenfunctions

$$\Psi_j = \begin{pmatrix} -b_j^* \\ 1 \end{pmatrix} \phi_j, \ \Psi_k = \begin{pmatrix} -b_k^* \\ 1 \end{pmatrix} \phi_k,$$

and co-eigenfunctions

$$\Psi_j^* = \frac{1}{1 + b_j b_j^*} \begin{pmatrix} -b_j \\ 1 \end{pmatrix} \phi_j, \ \Psi_k^* = \frac{1}{1 + b_k b_k^*} \begin{pmatrix} -b_k \\ 1 \end{pmatrix} \phi_k,$$

normalized so that $\langle \Psi_i^*, \Psi_m \rangle = \delta_{im}, i, m = j, k$.

We set the decompositions $Y = N(L_0) \oplus R(L_0)$ and $X = N(L_0) \oplus X_1$, where $N(L_0)$ is given by (3.6) and $X_1 = X \cap R(L_0)$. Define the operator P on Y by

$$PU = \langle U, \Phi_j^* \rangle \Phi_j + \langle U, \Phi_k^* \rangle \Phi_k.$$

Then $R(P) = N(L_0)$, and it is easy to check that $P^2 = P$ which implies that P is the projection onto $N(L_0)$. Thus, $Q = I - P$ is a projection onto $R(L_0)$ in Y. These result in that the system (3.4) is determined by a pair of equations

$$PF(w,\lambda)=0, \quad QF(w,\lambda)=0. \tag{3.7}$$

According to the decomposition of X, we rewrite $w \in X$ as the form $w = s\Phi_j + \tau\Phi_k + W$, where $(s,\tau) \in R^2$ and $W \in X_1$. By the second equation of (3.7), it follows from the implicit theorem that there exists a unique smooth function $W(s,\tau,\lambda) := W(s\Phi_j + \tau\Phi_k, \lambda)$ such that $W(0,0,0) = 0$ and $QF(s\Phi_j + \tau\Phi_k + W(s,\tau,\lambda),\lambda) = 0$ near the origin. It is easy to see that $W(0,0,\lambda) \equiv 0$ and then $W_\lambda(0,0,0) = W_{\lambda\lambda}(0,0) = \cdots = 0$. Substituting $W(s,\tau,\lambda)$ into the first equation of (3.7), we obtain

$$PF(s\Phi_j + \tau\Phi_k + W(s,\tau,\lambda),\lambda) = 0.$$

Let us write $L = L_0 + \delta\lambda M$, where $M = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\partial^2}{\partial x^2} \end{pmatrix}$. Then, according to the definition of P, the zeros of (3.4) are in one-to-one correspondence with the zeros of the reduced equation

$$\begin{pmatrix} \zeta(s,\tau,\lambda) \\ \vartheta(s,\tau,\lambda) \end{pmatrix} := \begin{pmatrix} \langle \Phi_j^*, H(s\Phi_j + \tau\Phi_k + W(s,\tau,\lambda),\lambda)\rangle \\ \langle \Phi_k^*, H(s\Phi_j + \tau\Phi_k + W(s,\tau,\lambda),\lambda)\rangle \end{pmatrix} = 0, \tag{3.8}$$

where $H(w,\lambda) := \delta\lambda M w + N(w)$, and so (3.4) has the solution as $w = s\Phi_j + \tau\Phi_k + W(s\Phi_j + \tau\Phi_k, \lambda)$ when s, τ and λ are solved by (3.8). Then we only need to consider the solvability of (3.8). It follows from [21] that the Eq. (3.4) with non-flux boundary conditions inherits a symmetric structure, and the above reduced form is rewritten as

$$\begin{pmatrix} \zeta(s,\tau,\lambda) \\ \vartheta(s,\tau,\lambda) \end{pmatrix} = \begin{pmatrix} sp(\bar{s},\bar{\tau},\lambda) + s^{k-1}\tau^j\beta(\bar{s},\bar{\tau},\lambda) \\ \tau q(\bar{s},\bar{\tau},\lambda) + s^k\tau^{j-1}\gamma(\bar{s},\bar{\tau},\lambda) \end{pmatrix}, \quad \bar{s} = s^2, \; \bar{\tau} = \tau^2. \tag{3.9}$$

Due to $H(0,\lambda) = 0$, it is easy to see that $\zeta_{00n} = \vartheta_{00n} = 0$, $n = 1, 2, \ldots$. However, the key of the further calculations for the Taylor coefficients of $\zeta(s,\tau,\lambda)$ and $\vartheta(s,\tau,\lambda)$ at the origin lies in the derivatives of W, which are determined by the second equation of (3.7). Clearly, we have $W_s(0,0,0) = 0$ and $W_\tau(0,0,0) = 0$, which lead to $\zeta_{100} = \vartheta_{100} = \zeta_{010} = \vartheta_{010} = 0$.

By straightforward calculations, the second derivatives of H at the origin are given as

$$\frac{\partial^2 H}{\partial s_i \partial\lambda} = \delta M\Phi_i, \quad \frac{\partial^2 H}{\partial s_i \partial s_m} = d^2N(\Phi_i,\Phi_m), \quad i,m = j,k \tag{3.10}$$

for $s_j = s$, $s_k = \tau$, where

$$d^2N(\Phi_i,\Phi_m) = \frac{1}{(1+\alpha^2)^2}\Big[(\alpha^2-1)(\Phi_{i1}\Phi_{m2} + \Phi_{i2}\Phi_{m1}) + 2\alpha(3-\alpha^2)\Phi_{i1}\Phi_{m1}\Big]\begin{pmatrix} 4 \\ \delta \end{pmatrix}. \tag{3.11}$$

According to (3.8)–(3.11), we omit the details of computations here and obtain

$$
\begin{aligned}
p_{001} &= \langle \Phi_j^*, \delta M \Phi_j \rangle = \frac{-\lambda_j b_j \bar{b}_j^*}{1 + b_j b_j^*} > 0, \quad q_{001} = \langle \Phi_k^*, \delta M \Phi_k \rangle = \frac{-\lambda_k b_k \bar{b}_k^*}{1 + b_k b_k^*} > 0, \\
\beta_0 &= \langle \Phi_j^*, d^2 N(\Phi_j, \Phi_k) \rangle = \begin{cases} \dfrac{(4 + \bar{b}_j^*) e_3}{\sqrt{2l}(1 + b_j b_j^*)}, & k = 2j, \\ 0, & k \neq 2j, \end{cases} \\
\gamma_0 &= \frac{1}{2} \langle \Phi_k^*, d^2 N(\Phi_j, \Phi_j) \rangle = \begin{cases} \dfrac{(4 + \bar{b}_k^*) e_1}{2\sqrt{2l}(1 + b_k b_k^*)}, & k = 2j, \\ 0, & k \neq 2j, \end{cases}
\end{aligned}
\tag{3.12}
$$

where

$$
\begin{aligned}
e_1 &= \frac{2[(\alpha^2 - 1) b_j + \alpha(3 - \alpha^2)]}{(1 + \alpha^2)^2} = \frac{\bar{e}_1}{2\alpha(1 + \alpha^2)}, \; \bar{e}_1 := 5 + \lambda_j - (1 + \lambda_j)\alpha^2, \\
e_2 &= \frac{2[(\alpha^2 - 1) b_k + \alpha(3 - \alpha^2)]}{(1 + \alpha^2)^2} = \frac{\bar{e}_2}{2\alpha(1 + \alpha^2)}, \; \bar{e}_2 := 5 + \lambda_k - (1 + \lambda_k)\alpha^2, \\
e_3 &= \frac{e_1 + e_2}{2} = \frac{\bar{e}_3}{4\alpha(1 + \alpha^2)}, \; \bar{e}_3 := \bar{e}_1 + \bar{e}_2, \\
4 &+ \bar{b}_j^* = \frac{20}{5 + \lambda_k} > 0, \;\; 4 + \bar{b}_k^* = \frac{20}{5 + \lambda_j} > 0.
\end{aligned}
$$

In order to seek the third order Taylor coefficients of $\zeta(s, \tau, \lambda)$ and $\vartheta(s, \tau, \lambda)$, the third derivatives of H at the origin are exhibited below.

$$
\begin{aligned}
H_{300} &= \frac{1}{2} d^2 N(\Phi_j, W_{ss}) + \frac{1}{3!} d^3 N(\Phi_j^3), \\
H_{030} &= \frac{1}{2} d^2 N(\Phi_k, W_{\tau\tau}) + \frac{1}{3!} d^3 N(\Phi_k^3), \\
H_{210} &= \frac{1}{2} d^2 N(\Phi_k, W_{ss}) + d^2 N(\Phi_j, W_{s\tau}) + \frac{1}{2} d^3 N(\Phi_j^2, \Phi_k), \\
H_{120} &= \frac{1}{2} d^2 N(\Phi_j, W_{\tau\tau}) + d^2 N(\Phi_k, W_{s\tau}) + \frac{1}{2} d^3 N(\Phi_j, \Phi_k^2), \\
H_{201} &= \frac{1}{2} d H_\lambda(W_{ss}) + d^2 N(\Phi_j, W_{s\lambda}),
\end{aligned}
\tag{3.13}
$$

where $H_{ijk} := \frac{1}{i!j!k!} \frac{\partial^{i+j+k}}{\partial s^i \partial \tau^j \partial \lambda^k} H(0, 0)$, the second Fréchet derivative of N is given as (3.11) and the third derivative is shown as

$$\begin{aligned} & d^3N(\Phi_i, \Phi_m, \Phi_n) \\ & \quad = \frac{1}{(1+\alpha^2)^3}\Big[2\alpha(3-\alpha^2)(\Phi_{i1}\Phi_{m1}\Phi_{n2} + \Phi_{i1}\Phi_{m2}\Phi_{n1} + \Phi_{i2}\Phi_{m1}\Phi_{n1}) \\ & \qquad + 6(\alpha^4 - 6\alpha^2 + 1)\Phi_{i1}\Phi_{m1}\Phi_{n1}\Big]\begin{pmatrix}4\\ \delta\end{pmatrix}. \end{aligned}$$

Moreover, from the second equation of (3.7), the second derivatives of W at the origin are determined by

$$W_{s_i s_m} = -L_0^{-1}Qd^2N(\Phi_i, \Phi_M), \quad W_{s_i\lambda} = -L_0^{-1}Q\delta M\Phi_i, \ i, m = j, k \tag{3.14}$$

for $s_j = s, s_k = \tau$.

From (3.14), we know that

$$L_0W_{ss} = -Qd^2N(\Phi_j, \Phi_j) = \begin{cases} -\frac{e_1}{l}\Big[\begin{pmatrix}4\\ \delta\end{pmatrix} + \langle cos\frac{2\pi j}{l}x\begin{pmatrix}4\\ \delta\end{pmatrix}, \Psi_k^*\rangle\Psi_k\Big], & k = 2j, \\ -\frac{e_1}{l}(1 + cos\frac{2\pi j}{l}x)\begin{pmatrix}4\\ \delta\end{pmatrix}, & k \neq 2j. \end{cases}$$

Let $W_{ss} = \sum_{i=0}^{\infty}\begin{pmatrix}a_i\\ b_i\end{pmatrix}\phi_i$, then we get

$$L_0W_{ss} = \sum_{i=0}^{\infty} B_i\begin{pmatrix}a_i\\ b_i\end{pmatrix}\phi_i,$$

where B_i is given by (3.5). Thus we obtain

$$\begin{aligned} W_{ss} &= \begin{cases} -\frac{e_1}{l}\Big[B_0^{-1}\begin{pmatrix}4\\ \delta\end{pmatrix} + \mu_k^{-1}\Big\langle cos\frac{2\pi j}{l}x\begin{pmatrix}4\\ \delta\end{pmatrix}, \Psi_k^*\Big\rangle\Psi_k\Big], & k = 2j, \\ -\frac{e_1}{l}\Big(B_0^{-1} + B_{2j}^{-1}cos\frac{2\pi j}{l}x\Big)\begin{pmatrix}4\\ \delta\end{pmatrix}, & k \neq 2j, \end{cases} \\ &= \begin{cases} -\frac{e_1}{5lg_1}\Big[\begin{pmatrix}0\\ 5\end{pmatrix} + \frac{5(\zeta_{jk}-5)}{\zeta_{jk}(1+b_kb_k^*)}\begin{pmatrix}-b_k^*\\ 1\end{pmatrix}cos\frac{\pi k}{l}x\Big], & k = 2j, \\ -\frac{e_1}{5lg_1}\Big[\begin{pmatrix}0\\ 5\end{pmatrix} + \frac{\lambda_k}{3(4\lambda_j - \lambda_k)}\begin{pmatrix}16d_j\lambda_j\\ 5+4\lambda_j\end{pmatrix}cos\frac{2\pi j}{l}x\Big], & k \neq 2j, \end{cases} \end{aligned} \tag{3.15}$$

where we use that $\dfrac{\delta - 4b_k}{\mu_k} = \dfrac{\zeta_{jk} - 5}{g_1\zeta_{jk}}$, $\zeta_{jk} := (5+\lambda_j)(1+b_k b_k^*)$. As the argument above, by the Fourier expansion for $W_{\tau\tau}$ and $W_{s\tau}$, we have

$$W_{\tau\tau} = -\frac{e_2}{5lg_1}\Big[\begin{pmatrix}0\\5\end{pmatrix} + \frac{\lambda_j}{3(4\lambda_k - \lambda_j)}\begin{pmatrix}16d_j\lambda_k\\5+4\lambda_k\end{pmatrix} cos\frac{2\pi k}{l}x\Big]. \tag{3.16}$$

$$W_{s\tau} = \begin{cases} -\dfrac{e_3}{5lg_1}\Big[\lambda_+ \begin{pmatrix}4d_j\lambda_{j+k}\\5+\lambda_{j+k}\end{pmatrix} cos\dfrac{\pi(j+k)}{l}x + \dfrac{5(\zeta_{kj}-5)}{\zeta_{kj}(1+b_j b_j^*)}\begin{pmatrix}-b_j^*\\1\end{pmatrix} cos\dfrac{\pi j}{l}x\Big], & k = 2j, \\ -\dfrac{e_3}{5lg_1}\Big[\lambda_+ \begin{pmatrix}4d_j\lambda_{j+k}\\5+\lambda_{j+k}\end{pmatrix} cos\dfrac{\pi(j+k)}{l}x + \lambda_- \begin{pmatrix}4d_j\lambda_{k-j}\\5+\lambda_{k-j}\end{pmatrix} cos\dfrac{\pi(k-j)}{l}x\Big], & k \neq 2j, \end{cases} \tag{3.17}$$

where $\bar{\lambda}_+ := \dfrac{1}{(\lambda_j - \lambda_{j+k})(\lambda_k - \lambda_{j+k})}$, $\bar{\lambda}_- := \dfrac{1}{(\lambda_j - \lambda_{k-j})(\lambda_k - \lambda_{k-j})}$, $\lambda_+ = \bar{\lambda}_+\lambda_j\lambda_k$, $\lambda_- = \bar{\lambda}_-\lambda_j\lambda_k$, $\dfrac{\delta - 4b_j}{\mu_j} = \dfrac{\zeta_{kj} - 5}{g_1\zeta_{kj}}$.

Combining (3.13) and (3.15)–(3.17), straightforward calculations yield the Taylor coefficients of $\zeta(s,\tau,\lambda)$ and $\vartheta(s,\tau,\lambda)$ as follows. It is obvious that

$$q_{010} = \langle \Phi_k^*, H_{030}\rangle = \Big\langle \Phi_k^*, \frac{1}{2}d^2N(\Phi_k, W_{\tau\tau})\Big\rangle + \Big\langle \Phi_k^*, \frac{1}{3!}d^3N(\Phi_k^3)\Big\rangle,$$

where the first and second term of q_{010} are respectively given by

$$q_{010}^1 = \frac{(4+\bar{b}_k^*)\bar{\Lambda}_{kj}}{40l\alpha^2(1+\alpha^2)^2(1+b_k b_k^*)},\ q_{010}^2 = \frac{(4+\bar{b}_k^*)\bar{\Lambda}_k}{40l\alpha^2(1+\alpha^2)^2(1+b_k b_k^*)},$$

$$\bar{\Lambda}_{kj} := \frac{5+\lambda_k - (1+\lambda_k)\alpha^2}{3(4\lambda_k - \lambda_j)}\Big[(\alpha^2-1)(120\lambda_k - 5\lambda_j + 8\lambda_j\lambda_k) + \frac{160\alpha^2(3-\alpha^2)}{1+\alpha^2}\Big],$$

$$\bar{\Lambda}_k := 15\alpha^2[(\alpha^2-3)\lambda_k + \alpha^2 - 11].$$

Therefore, we obtain

$$q_{010} = q_{010}^1 + q_{010}^2 = \frac{(4+\bar{b}_k^*)\bar{q}_{010}}{40l\alpha^2(1+\alpha^2)^2(1+b_k b_k^*)},\quad \bar{q}_{010} = \bar{\Lambda}_{kj} + \bar{\Lambda}_k. \tag{3.18}$$

In the same way, we have

$$q_{100} = \langle \Phi_k^*, H_{210}\rangle = q_{100}^1 + q_{100}^2 + q_{100}^3, \tag{3.19}$$

where

$$q^1_{100} = -\frac{e_1(4+\bar{b}^*_k)\Lambda}{2lg_1(1+\alpha^2)^2(1+b_k b^*_k)}, \quad \Lambda = \alpha^2 - 1,$$

$$q^2_{100} = \begin{cases} -\dfrac{e_3(4+\bar{b}^*_{2j})\tilde{q}^2_{100}}{10lg_1(1+\alpha^2)^2(1+b_{2j}b^*_{2j})}, \tilde{q}^2_{100} = \tilde{\Lambda}_j + \tilde{\Lambda}_{jkj}, & k = 2j, \\ -\dfrac{e_3(4+\bar{b}^*_k)\bar{q}^2_{100}}{10lg_1(1+\alpha^2)^2(1+b_k b^*_k)}, \bar{q}^2_{100} = \bar{\lambda}_+\bar{\Lambda}_{j(j+k)k} + \bar{\lambda}_-\bar{\Lambda}_{j(k-j)k}, & k \neq 2j, \end{cases}$$

$$\tilde{\Lambda}_j := \frac{1}{5}\left[(\alpha^2-1)(25+9\lambda_j) + 36\alpha(3-\alpha^2)d_j\lambda_j\right],$$

$$\tilde{\Lambda}_{imn} := \frac{5(\zeta_{mi}-5)}{\zeta_{mi}(1+b_i b^*_i)}\left[(\alpha^2-1)(1-b_n b^*_i) - 2\alpha(3-\alpha^2)b^*_i\right],$$

$$\bar{\Lambda}_{imn} := (\alpha^2-1)[5\lambda_i + (5+2\lambda_i)\lambda_m]\lambda_n + \frac{40\alpha^2(3-\alpha^2)}{1+\alpha^2}\lambda_m,$$

$$q^3_{100} = \frac{(4+\bar{b}^*_k)\bar{q}^3_{100}}{60l\alpha^2(1+\alpha^2)^2(1+b_k b^*_k)}, \ \bar{q}^3_{100} = 2\bar{\Lambda}_j + \bar{\Lambda}_k,$$

$$p_{010} = \langle \Phi^*_k, H_{120}\rangle = p^1_{010} + p^2_{010} + p^3_{010}, \tag{3.20}$$

where

$$p^1_{010} = -\frac{e_2(4+\bar{b}^*_j)\Lambda}{2lg_1(1+\alpha^2)^2(1+b_j b^*_j)},$$

$$p^2_{010} = \begin{cases} -\dfrac{e_3(4+\bar{b}^*_j)\tilde{p}^2_{010}}{10lg_1(1+\alpha^2)^2(1+b_j b^*_j)}, \tilde{p}^2_{010} = \tilde{\Lambda}_j - \tfrac{27}{8}\Lambda + \tilde{\Lambda}_{jkk}, & k = 2j, \\ -\dfrac{e_3(4+\bar{b}^*_j)\bar{p}^2_{010}}{10lg_1(1+\alpha^2)^2(1+b_j b^*_j)}, \bar{p}^2_{010} = \bar{\lambda}_+\bar{\Lambda}_{k(j+k)j} + \bar{\lambda}_-\bar{\Lambda}_{k(k-j)j}, & k \neq 2j, \end{cases}$$

$$p^3_{010} = \frac{(4+\bar{b}^*_j)\bar{p}^3_{010}}{60l\alpha^2(1+\alpha^2)^2(1+b_j b^*_j)}, \ \bar{p}^3_{010} = 2\bar{\Lambda}_k + \bar{\Lambda}_j,$$

and

$$p_{100} = \langle \Phi^*_j, H_{300}\rangle = \begin{cases} \dfrac{(4+\bar{b}^*_j)\tilde{p}_{100}}{40l\alpha^2(1+\alpha^2)^2(1+b_j b^*_j)}, \tilde{p}_{100} = \bar{e}_1(10\Lambda + \tilde{\Lambda}_{kjj}) + \bar{\Lambda}_j, & k = 2j, \\ \dfrac{(4+\bar{b}^*_j)\bar{p}_{100}}{40l\alpha^2(1+\alpha^2)^2(1+b_j b^*_j)}, \bar{p}_{100} = \bar{\Lambda}_{jk} + \bar{\Lambda}_j, & k \neq 2j. \end{cases} \tag{3.21}$$

Table 1 $(j, k) = (1, 2)$: normal forms $(\tilde{\zeta}, \tilde{\vartheta})$

Case	$\tilde{p}$	$\tilde{q}$
(1)	$-s(\tau + \varepsilon_5 \lambda^2)$	$\tau(\varepsilon_3 \tau^2 + \varepsilon_4 \lambda) + \varepsilon_2 s^2$
(2)	$-s(\tau + \varepsilon_1 \lambda)$	$\tau(\varepsilon_3 \tau^2 + \kappa \lambda^2) + \varepsilon_2 s^2$
(3)	$-s(\varepsilon_5 \tau^2 + \rho \lambda)$	$\tau(\varepsilon_3 \tau^2 + \varepsilon_4 \lambda) + \varepsilon_2 s^2$
(4)	$-s(\tau + \varepsilon_1 \lambda)$	$\tau(\varepsilon_3 \tau^2 + \varepsilon_4 \lambda) + \varepsilon_5 s^4 + \varsigma s^2 \lambda$
(5)	$-s(\tau + \varepsilon_1 \lambda)$	$\tau(\varepsilon_5 \tau^4 + \varepsilon_4 \lambda) + \varepsilon_2 s^2$

According to [21], there are two different cases such as $(j, k) = (1, 2)$ and $j > 1, k > j$ for the solvability of (3.8). Here we only deal with the former case below to present a complete analysis based on (3.18)–(3.21).

For the case $(j, k) = (1, 2)$, we have $\alpha^2 = \dfrac{5 + 5\lambda_1 + \frac{4}{5}\lambda_1^2}{3 - 5\lambda_1 - \frac{4}{5}\lambda_1^2}$ from (3.1). Noting that the fact $\lambda_{12}^* < 3$ implies $\lambda_1 < \iota^*$, with the value ι^* determined numerically to be about $\iota^* = 0.5514$. Therefore, the further discussions are based on $\lambda_1 = (\pi/l)^2 \in (0, \iota^*)$.

From [21], if $p_{001} q_{001} \beta_0 \gamma_0 q_{010} \neq 0$, then the reduced Eq. (3.9) is equivalent to the normal form

$$\begin{pmatrix} -s(\tau + \varepsilon_1 \lambda) \\ \tau(\varepsilon_3 \tau^2 + \varepsilon_4 \lambda) + \varepsilon_2 s^2 \end{pmatrix} \tag{3.22}$$

where

$$\varepsilon_1 = -\mathrm{sgn} p_{001}, \ \varepsilon_2 = -\mathrm{sgn}(\beta_0 \gamma_0), \ \varepsilon_3 = \mathrm{sgn} q_{010}, \ \varepsilon_4 = \mathrm{sgn} q_{001}.$$

For $p_{001}, q_{001}, \beta_0, \gamma_0, q_{010} = 0$, five different normal forms summarized in Table 1 are possible, and the conditions for the normal forms are shown in Tables 2 and 3.

In view of (3.22), Tables 1 and 2, we further give the detailed discussions. Obviously, for our model, it is always valid that $p_{001} > 0$ and $q_{001} > 0$, which lead to $\varepsilon_1 = -1$ and $\varepsilon_4 = 1$. According to (3.12) and (3.18), we find that $\beta_0, \gamma_0, q_{010}$ have the same sign with $\bar{e}_3, \bar{e}_1, \bar{q}_{010}$, respectively, which are not equal to zero simultaneously. Moreover, $\bar{e}_3 = 0, \bar{e}_1 = 0$, and $\bar{q}_{010} = 0$ are all $\lambda_1' s$ equations, and then by a simple monotonicity analysis, it is readily found that $0 < \iota_0 < \iota_1 < \iota_2 < \iota_3 < \iota^*$, shown in Fig. 2, where ι_1, ι_2 is the corresponding root of $\bar{e}_3 = 0, \bar{e}_1 = 0$, and ι_0, ι_3 are the roots of $\bar{q}_{010} = 0$. The value of $\iota_0, \iota_1, \iota_2, \iota_3$ can be determined numerically to be 0.0355, 0.2366, 0.2762, 0.3908, respectively. Thus combining Fig. 2 with (3.22), we have the following result.

Theorem 3.1 *If* $\lambda_1 \neq \iota_i, i = 0, 1, 2, 3$, *then the reduced problem* (3.8) *is equivalent to the normal form*

$$\begin{cases} -s(\tau - \lambda) = 0, \\ \tau(\varepsilon_3 \tau^2 + \lambda) + \varepsilon_2 s^2 = 0, \end{cases}$$

Table 2 $(j, k) = (1, 2)$: conditions for the normal forms of Table 1

Case	Zero	Non-zero	$\varepsilon_5 = \text{sgn}(.)$
(1)	p_{001}	p_{002}	$-p_{002}$
(2)	q_{001}	$q_{002}, q_{010}p_{001}^2 + q_{002}\beta_0^2$	–
(3)	β_0	$p_{010}, p_{010}q_{001} - p_{001}q_{010}$	$-p_{010}$
(4)	γ_0	D_4, D_5	$-D_4$
(5)	q_{010}	q_{020}	q_{020}

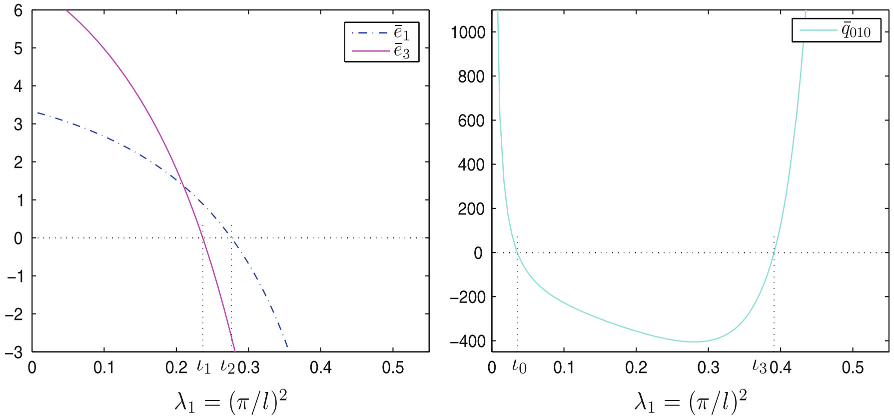

Fig. 2 The zeros of $\bar{e}_1$, $\bar{e}_3$ and $\bar{q}_{010}$

where $\varepsilon_3 = \begin{cases} +, & \lambda_1 \in (0, \iota_0) \cup (\iota_3, \iota^*), \\ -, & \lambda_1 \in (\iota_0, \iota_3) \end{cases}$ *and* $\varepsilon_2 = \begin{cases} -, & \lambda_1 \in (0, \iota_1) \cup (\iota_2, \iota^*), \\ +, & \lambda_1 \in (\iota_1, \iota_2). \end{cases}$

Case 1 $\lambda_1 = \iota_1$ i.e. $\beta_0 = 0$

It is clear that $\gamma_0 > 0$ from Fig. 2 and $\bar{e}_3 = 0$ from (3.12), leading to $W_{s\tau} = 0$ and $p_{010}^2 = 0$. Thus, by (3.20) and (3.18), we obtain

$$p_{010} = \frac{(4 + \bar{b}_1^*)\hat{p}_{010}}{4l\alpha^2(1 + \alpha^2)^2(1 + b_1 b_1^*)} < 0,$$

$$\hat{p}_{010} = (5\alpha^4 - 19\alpha^2 - 4)\lambda_1 + 2\alpha^4 - 27\alpha^2 - 5 < 0,$$

$$q_{010} = \frac{(4 + \bar{b}_2^*)\hat{q}_{010}}{24l\alpha^2(1 + \alpha^2)^2(1 + b_2 b_2^*)} < 0,$$

$$\hat{q}_{010} = 4(4\alpha^4 - 17\alpha^2 - 5)\lambda_1 + 4\alpha^4 - 69\alpha^2 - 25 < 0.$$

Then $\rho \approx -0.5662$ determined by numerical computation, and

$$p_{010}q_{001} - p_{001}q_{010} = \frac{-4\lambda_1^3[3(5 + 4\lambda_1)\hat{p}_{010} - 2(5 + \lambda_1)\hat{q}_{010}]}{15lf_1 g_0 \alpha^2(1 + \alpha^2)^2(1 + b_1 b_1^*)(1 + b_2 b_2^*)} > 0.$$

Table 3 The parameters appearing in Tables 1 and 2

$\kappa = \dfrac{q_{002}\beta_0^2}{p_{001}^2 \lvert q_{010}\rvert}$
$\rho = -\dfrac{p_{001}}{\lvert p_{010}\rvert}\left\lvert\dfrac{q_{010}}{q_{001}}\right\rvert$
$\varsigma = -D_5\left\lvert\dfrac{q_{001}}{p_{001}D_4}\right\rvert^{\frac{1}{2}}$
$D_4 = \beta_0\gamma_{100} - p_{100}q_{100}$
$D_5 = \beta_0\gamma_{001} - p_{100}q_{001} - p_{001}q_{100}$

Therefore, according to Tables 1 and 2, we have the following result for $\lambda_1 = \iota_1$.

Theorem 3.2 *If $\lambda_1 = \iota_1$, then the reduced problem* (3.8) *is equivalent to the normal form*

$$\begin{cases} -s(\tau^2 + \rho\lambda) = 0, \\ \tau(-\tau^2 + \lambda) + s^2 = 0. \end{cases}$$

Case 2 $\lambda_1 = \iota_2$ i.e. $\gamma_0 = 0$

For this case, it is readily apparent that $\bar{e}_1 = 0$, which results in $\bar{e}_3 = \bar{e}_2 = 3\lambda_1(1-\alpha^2) < 0$, $W_{ss} = 0$ and $q^1_{100} = 0$. Thus, from (3.21) and (3.19), we have

$$p_{100} = \frac{3(4+\bar{b}_1^*)}{8l(1+\alpha^2)^2(1+b_1b_1^*)}[(\alpha^2-3)\lambda_1 + \alpha^2 - 11],$$

$$q_{100} = \frac{3(4+\bar{b}_2^*)}{8l\alpha^2(1+\alpha^2)^2(1+b_2b_2^*)} \left\{\left(\frac{5}{\zeta_{21}} - \frac{1}{5}\right)(\alpha^2-1)^2\lambda_1 + 2\alpha^2\left[2(\alpha^2-3)\lambda_1 + \alpha^2 - 11\right]\right\}.$$

In view of Tables 1 and 2, it is very natural to further estimate D_4 and D_5. By the Fourier expansion as above, from (3.14) we have

$$W_{s\lambda} = \frac{-\lambda_j b_j^2}{g_0(1+b_jb_j^*)^2}\begin{pmatrix}-b_j^* \\ 1\end{pmatrix}\phi_j.$$

Then we have

$$\gamma_{001} = \langle \Phi_2^*, d^2N(\Phi_1, W_{s\lambda})\rangle = \frac{-\lambda_1 b_1^2(\alpha^2-1)(4+\bar{b}_2^*)}{\sqrt{2l}\,g_0(1+\alpha^2)^2(1+b_1b_1^*)(1+b_2b_2^*)}.$$

Due to $W_{ss} = 0$, from the second equation of (3.7) we find that W_{sss} satisfies

$$L_0 W_{sss}(0,0,0) = -Qd^3 N(\Phi_j, \Phi_j, \Phi_j)$$
$$= -\frac{6e_4}{l\sqrt{2l}}\left[cos\frac{3\pi j}{l}x\begin{pmatrix}4\\ \delta\end{pmatrix} + \left\langle 3cos\frac{\pi j}{l}x\begin{pmatrix}4\\ \delta\end{pmatrix}, \Psi_j^*\right\rangle \Psi_j\right],$$
$$e_4 = \frac{1}{4(1+\alpha^2)^2}[(\alpha^2-3)\lambda_j + \alpha^2 - 11].$$

Thus, for $(j,k) = (1,2)$ we get

$$W_{sss}(0,0,0) = -\frac{6e_4}{5l\sqrt{2l}g_1}\left\{\frac{1}{10}\begin{pmatrix}36d_1\lambda_1\\ 5+9\lambda_1\end{pmatrix}cos\frac{3\pi}{l}x + \frac{15(\zeta_{21}-5)}{\zeta_{21}(1+b_1b_1^*)}\begin{pmatrix}-b_1^*\\ 1\end{pmatrix}cos\frac{\pi}{l}x\right\}. \tag{3.23}$$

It is straightforward to calculate that

$$d^4N(\Phi_i,\Phi_m,\Phi_n,\Phi_l) = \frac{6}{(1+\alpha^2)^4}\Big[(\alpha^4-6\alpha^2+1)(\Phi_{i1}\Phi_{m1}\Phi_{n1}\Phi_{l2} + \Phi_{i1}\Phi_{m1}\Phi_{n2}\Phi_{l1} + \Phi_{i1}\Phi_{m2}\Phi_{n1}\Phi_{l1} + \Phi_{i2}\Phi_{m1}\Phi_{n1}\Phi_{l1}) - 4\alpha(\alpha^4-10\alpha^2+5)\Phi_{i1}\Phi_{m1}\Phi_{n1}\Phi_{l1}\Big]\begin{pmatrix}4\\ \delta\end{pmatrix}. \tag{3.24}$$

Then combining (3.23) and (3.24), we obtain

$$\gamma_{100} = \left\langle \Phi_2^*, \frac{1}{3!}d^2N(\Phi_1, W_{sss}) + \frac{1}{4!}d^4N(\Phi_1^4)\right\rangle$$
$$= \frac{4+\bar{b}_2^*}{4(\sqrt{2l})^3\alpha(1+\alpha^2)^3(1+b_2b_2^*)}\Big\{[(\alpha^2-3)\lambda_1+\alpha^2-11](\alpha^2-1)\left(\frac{11}{5}-\frac{15}{\zeta_{21}}\right) - 4[(\alpha^4-6\alpha^2+1)\lambda_1+\alpha^4-18\alpha^2+5]\Big\}.$$

Therefore, we have

$$D_4 = \frac{6(4+\bar{b}_1^*)(4+\bar{b}_2^*)\hat{D}_4}{5(8l\alpha)^2(1+\alpha^2)^4(1+b_1b_1^*)(1+b_2b_2^*)} < 0,$$
$$D_5 = \frac{6\lambda_1^3\hat{D}_5}{25lf_1g_0\alpha^2(1+\alpha^2)^2(1+b_1b_1^*)(1+b_2b_2^*)} > 0,$$
$$\hat{D}_4 = 10\lambda_1(\alpha^2-1)[(\alpha^4-6\alpha^2+1)\lambda_1+\alpha^4-18\alpha^2+5] - [(\alpha^2-3)\lambda_1+\alpha^2-11][(34\alpha^4-98\alpha^2+4)\lambda_1+15\alpha^2(\alpha^2-11)] < 0,$$
$$\hat{D}_5 = 5\alpha^2(45+12\lambda_1)[(\alpha^2-3)\lambda_1+\alpha^2-11] + 4\lambda_1(5+\lambda_1)(9\alpha^4-28\alpha^2-1) < 0.$$

$$\varsigma = -\frac{6(5+4\lambda_1)\lambda_1^3 \hat{D}_5}{25 f_1 g_0 \alpha (1+b_2 b_2^*)} \left| \frac{5+4\lambda_1}{30(5+\lambda_1)\hat{D}_4} \right|^{\frac{1}{2}} < 0.$$

Theorem 3.3 *If $\lambda_1 = \iota_2$, then the reduced problem (3.8) is equivalent to the normal form*

$$\begin{cases} -s(\tau - \lambda) = 0, \\ \tau(-\tau^2 + \lambda) + s^4 + \varsigma s^2 \lambda = 0. \end{cases}$$

Remark 3.4 For the case $\lambda_1 = \iota_0$ or ι_3, i.e. $q_{010} = 0$, we further need to consider the term q_{020}, but the calculation of which is more tedious and omitted here. The preliminary analysis shows that $q_{020} \neq 0$ for $\lambda_1 = \iota_0$ or ι_3, and so the reduced problem (3.8) is equivalent to the normal form

$$\begin{cases} -s(\tau - \lambda) = 0, \\ \tau(\pm\tau^4 + \lambda) + s^2 = 0. \end{cases}$$

Therefore, we have a complete analysis for the case $(j, k) = (1, 2)$, which means that no further equivalence can exist, that is to say, (3.8) is solvable in this case. But for the case $j > 1, k > j$, one can obtain a detailed result for the specific j and k from [21] and (3.18)–(3.21), which is omitted here.

In a word, the discussions above illustrate that the reduced problem (3.8) can be solved by equivalence. Thus, for some $d_j = d_k$ $(j \neq k)$, the original system (2.1) has the solution as the following form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^* \\ v^* \end{pmatrix} + s \begin{pmatrix} 1 \\ b_j \end{pmatrix} \phi_j + \tau \begin{pmatrix} 1 \\ b_k \end{pmatrix} \phi_k + W(s, \tau, \lambda), \tag{3.25}$$

when (s, τ), close to $(0, 0)$, is the local zero of (3.8), and here $W(s, \tau, \lambda)$ satisfies $W(0, 0, \lambda) = 0$, $W_\lambda(0, 0, 0) = W_{\lambda\lambda}(0, 0) = \cdots = 0$, $W_s(0, 0, 0) = 0$ and $W_\tau(0, 0, 0) = 0$. The solution form is characterized by two distinct modes such as ϕ_j and ϕ_k, and what we discuss here is suitable for any double bifurcation problem, not just the first one. In particular, if for some integer j, we have

$$\alpha^2 = \frac{5 + \lambda^*_{j(j+1)}}{3 - \lambda^*_{j(j+1)}}$$

in (3.1), then the double bifurcation is the first bifurcation achieved at $d_j = d_{j+1}$.

4 Stability of bifurcation solutions

In this section, we focus on the stability of the simple and double bifurcation solutions given by (2.3) and (3.25). For convenience of the following discussion, we denote $d_m := \tilde{d} = \min_{1 \le i \le i_\alpha} d_i$ in (2.2).

Theorem 4.1 *Assume that $j \neq m$. Then L_0 has a positive eigenvalue, and both simple and double bifurcation solutions are unstable.*

Proof Recall that the linearized operator of the steady state system of (2.1) evaluated at (d_j, u^*, v^*) is given by

$$L_0 = \begin{pmatrix} \Delta + f_0 & f_1 \\ \delta g_0 & \delta(d_j \Delta + g_1) \end{pmatrix}, \quad \Delta = \frac{\partial^2}{\partial x^2}.$$

In the following, we begin to discuss the eigenvalue of L_0. Suppose that μ is an eigenvalue of L_0 with a corresponding eigenfunction $(\phi(x), \psi(x))$. Then we have

$$\frac{\partial^2 \phi}{\partial x^2} + (f_0 - \mu)\phi + f_1 \psi = 0, \quad \delta d_j \frac{\partial^2 \psi}{\partial x^2} + \delta g_0 \phi + (\delta g_1 - \mu)\psi = 0.$$

By the Fourier expansion $\phi = \sum_{i=0}^{\infty} a_i \phi_i$, $\psi = \sum_{i=0}^{\infty} b_i \phi_i$, we obtain

$$\sum_{i=0}^{\infty} \begin{pmatrix} f_0 - \lambda_i - \mu & f_1 \\ \delta g_0 & \delta(g_1 - d_j \lambda_i) - \mu \end{pmatrix} \begin{pmatrix} a_i \\ b_i \end{pmatrix} \phi_i = 0.$$

It follows that the eigenvalues of L_0 are given by

$$\mu^2 - P_i(d_j)\mu + Q_i(d_j) = 0, \quad i = 0, 1, 2, \ldots,$$

where

$$P_i(d_j) = f_0 - \lambda_i + \delta(g_1 - d_j \lambda_i) = f_0 + \delta g_1 - (1 + \delta d_j)\lambda_i < 0 \tag{4.1}$$

by the condition (H), and

$$Q_i(d_j) = \delta[(f_0 - \lambda_i)(g_1 - d_j \lambda_i) - f_1 g_0] = \delta[d_j \lambda_i (\lambda_i - f_0) + \frac{\alpha}{1 + \alpha^2}(\lambda_i + 5)]. \tag{4.2}$$

When $j \neq m$, we find $Q_m(d_j) < 0$ from (4.2), and then L_0 has a positive eigenvalue. Thus, according to the perturbation theory of linearized operator, the bifurcation solutions described by (2.3) and (3.25) for $j \neq m$ are unstable. The proof is completed. □

Lemma 4.2 *Suppose that $j = m$ and $d_j \neq d_k$ for any integer $k \neq j$. Then 0 is a simple eigenvalue of L_0 with the largest real part, and all the other eigenvalues of L_0 lie in the left half complex plane.*

Proof For $j = m$, it follows from [18] that

$$N(L_0) = \text{span}\{\Phi_m\} \quad \text{and} \quad N(L_0^*) = \text{span}\{\Phi_m^*\},$$

where Φ_m, Φ_m^* are shown in (3.6) and

$$\langle \Phi_m, \Phi_m^* \rangle = 1 > 0,$$

which implies $\Phi_m \notin R(L_0)$ by the Fredholm alternative, and so 0 is a simple eigenvalue of L_0. From (4.1) and (4.2), we have that for all i, $P_i(d_m) < 0$ and

$$Q_m(d_m) = 0, \; Q_i(d_m) > 0, \quad i = 0, 1, 2, \ldots, m-1, m+1, \ldots.$$

Hence, 0 is a simple eigenvalue of L_0 with the largest real part, and all the other eigenvalues of L_0 lie in the left half complex plane. Thus we complete the proof. □

For the case d_j such that $d_j \neq d_k$ (any $k \neq j$), in the same way as Sect. 3, we take the decompositions $Y = N(L_0) \oplus R(L_0)$ and $X = N(L_0) \oplus X_1$, where $N(L_0) = \text{span}\{\Phi_j\}$ and $X_1 = X \cap R(L_0)$. We define the projection P on Y by

$$PU = \langle U, \Phi_j^* \rangle \Phi_j.$$

and set $w \in X$ in the form $w = s\Phi_j + W$, where $s \in R$ and $W \in X_1$. Following from the implicit theorem, $W(s, \lambda) := W(s\Phi_j, \lambda)$ defined near the origin is uniquely solvable from the second equation of (3.7). It is obvious that $W_s(0, 0) = 0$ and $W(0, \lambda) \equiv 0$, leading to $W_\lambda(0, 0) = W_{\lambda\lambda}(0, 0) = \cdots = 0$. Substituting $W(s, \lambda)$ into the first equation of (3.7), we obtain

$$PF(s\Phi_j + W(s, \lambda), \lambda) = 0. \tag{4.3}$$

As in Sect. 3, owing to the projection P, (4.3) can be rewritten in the following reduced equation

$$h(s, \lambda) = \langle \Phi_j^*, H(s\Phi_j + W(s, \lambda), \lambda) \rangle = 0. \tag{4.4}$$

Thus the zeros of (4.4) are corresponding to the simple bifurcation solutions of (2.1) shown in (2.3).

Obviously, $h_s(0, 0) = 0$ from $W_s(0, 0) = 0$ and $h_\lambda(0, 0) = h_{\lambda\lambda}(0, 0) = \cdots = 0$ from $H(0, \lambda) = 0$. Here, we notice that

$$W_{ss}(0, 0) = -\frac{e_1}{5lg_1}\left[\begin{pmatrix} 0 \\ 5 \end{pmatrix} + \frac{5(f_0 - \lambda_j)}{3(4\lambda_j^2 + 25\lambda_j - 5f_0)}\begin{pmatrix} 16d_j\lambda_j \\ 5 + 4\lambda_j \end{pmatrix} cos\frac{2\pi j}{l}x\right]$$

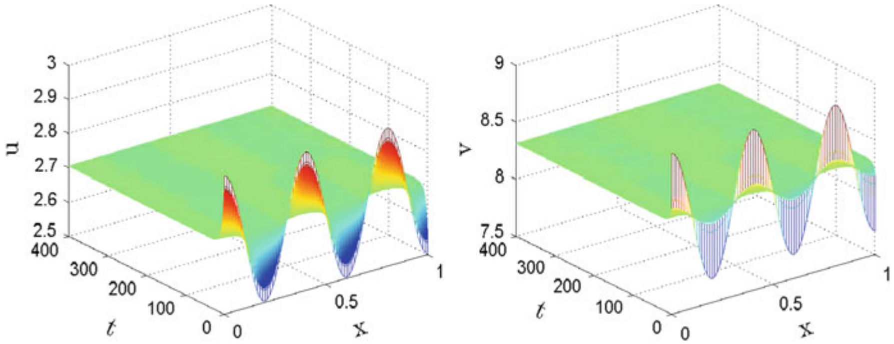

Fig. 3 The constant solution is asymptotically stable for $\delta = 9.7607, \alpha = 2.7034, l = 13.3$ and $d = 1.75 < \tilde{d}$ with the initial values $u_0(x) = u^* + 0.2cos(5\pi x), v_0(x) = v^* + 0.6cos(5\pi x)$

according to (3.14). Furthermore, we can get

$$
\begin{aligned}
h_{\lambda s}(0,0) &= \langle \Phi_j^*, \delta M \Phi_j \rangle = \frac{-\lambda_j b_j \bar{b}_j^*}{1 + b_j b_j^*} > 0, \;\; h_{ss}(0,0) = h_s^{(4)}(0,0) = 0, \\
h_{sss}(0,0) &= \langle \Phi_j^*, \frac{1}{2} d^2 N(\Phi_j, W_{ss}) + \frac{1}{3!} d^3 N(\Phi_j^3) \rangle \\
&= \frac{4 + \bar{b}_j^*}{8l\alpha^2(1+\alpha^2)^2(1+b_j b_j^*)} \left\{ \frac{\bar{e}_1}{3(4\lambda_j^2 + 25\lambda_j - 5f_0)} \left[(\alpha^2 - 1)(16\lambda_j^2 + 125\lambda_j \right.\right. \\
&\quad \left.\left. + 8\lambda_j f_0 - 5f_0) + \frac{32\alpha^2(3-\alpha^2)}{1+\alpha^2}(5+\lambda_j) \right] + 3\alpha^2[(\alpha^2 - 3)\lambda_j + \alpha^2 - 11] \right\}.
\end{aligned}
\tag{4.5}
$$

Therefore, combining Lemma 4.2 with page 320 of [19], we have the following stability result for the simple bifurcation solution $(u_j(s), v_j(s)), \;\; j = m$ in (2.3), where the stability theorem in [23] is not valid.

Theorem 4.3 *Suppose that $j = m$ and $d_j \neq d_k$ for any integer $k \neq j$. If $h_{sss}(0,0) < 0$ (> 0), then the bifurcation solution $(u_j(s), v_j(s))$ is stable (unstable) for both $s < 0$ and $s > 0$.*

Remark 4.4 The above discussion yields no information about the direction and number of simple bifurcation solutions. By [22], we know that $\lambda_s'(0) = 0$ for $h_{ss}(0,0) = 0$, and then no transcritical bifurcation occurs. When $h_{sss}(0,0) \neq 0$ in (4.5), (4.4) is equivalent to $\mathrm{sgn}(h_{sss}(0,0))s^3 + \lambda s$ from [19], and then a pitchfork bifurcation occurs for every simple bifurcation point in our system (2.1), specifically speaking, subcritical bifurcation for $h_{sss}(0,0) > 0$ and supercritical bifurcation for $h_{sss}(0,0) < 0$. In particular, for the subcritical case in the work [18], the bifurcation curve finally turn back based on Theorems 2.1 and 2.4.

Remark 4.5 If $h_{sss}(0,0) < 0$ (> 0), then (2.1) has two non-constant steady state solutions when $\lambda > 0$ (< 0) and no solutions when $\lambda < 0$ (> 0). For $h_{sss}(0,0) =$

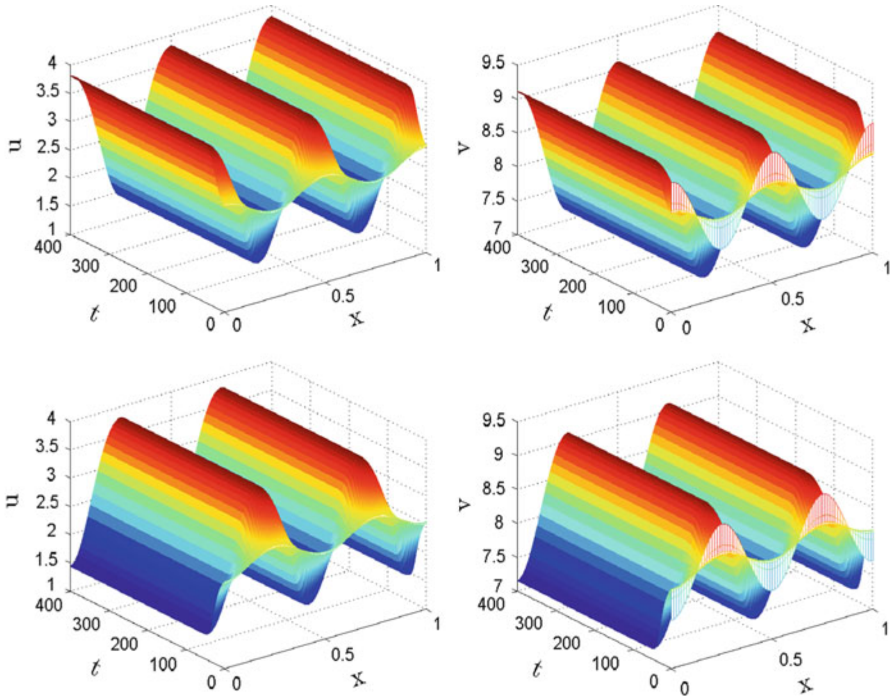

Fig. 4 The spatially nonhomogeneous steady states for $\delta = 9.7607$, $\alpha = 2.7034$, $l = 13.3$ and $d = 2.0 > d_4$. *Top* the initial values $u_0(x) = u^* + 0.2cos(4\pi x)$, $v_0(x) = v^* + 0.6cos(4\pi x)$; *bottom* the initial values $u_0(x) = u^* - 0.2cos(4\pi x)$, $v_0(x) = v^* - 0.6cos(4\pi x)$

0, when $h_s^{(5)}(0, 0) \neq 0$ of which the tedious calculation is omitted here, there are same results as $h_{sss}(0, 0) \neq 0$. Thus, the system (2.1) has at least two non-constant steady state solutions in a neighborhood of the simple bifurcation point d_j, which complements Theorem 2.4.

5 Numerical simulations

This section aims to illustrate the analytic results of the previous sections. The initial-boundary-value problem (2.1) is performed numerically by use of a standard implicit method, based on the Crank-Nicholson scheme. Here, we transform the spatial domain from $0 < x < l$ to $0 < \hat{x} < 1$ by putting $\hat{x} = x/l$, and still denote $\hat{x}$ by x in the numerical simulations.

We choose the same parameter values as for Fig. 1, namely $\alpha = 2.7034$ and $l = 13.3$, leading to that all the bifurcation points are $d_4 = 1.8768 < d_3 = d_5 = 2.3227 < d_2 = 4.1980 < d_1 = 14.8809 < d_6 = 39.8377$. This shows that $\tilde{d} = d_4$ (i.e. $m = 4$ in Sect. 4) and the first bifurcation point is $d_4 = 1.8768$. For these parameter values, the constant solution (u^*, v^*) is equal to $(2.7034, 8.3084)$. Taking $\delta = 9.7607$, Fig. 3 is devoted to demonstrate the constant solution is stable for $d = 1.75 < \tilde{d}$ which justifies the Lemma 2.2. In Fig. 4, for $d = 2.0 > d_4$, the stable spatially nonhomogeneous steady states with mode 4 form, as predicted in Theorem 4.3, here $h_{sss}(0, 0) < 0$ for $j = m = 4$ in (4.5).

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